Political Economics
Problem Set 1
Suggested solutions

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Note

The solution to this problem set should be uploaded to Studentportalen no later than **November 9** at 24:00. Write your individual answers on computer and put your name at the top of the document. This problem set is about Social choice, voting and bargaining. Please carefully motivate your answers. For any questions concerning the exercises, send me an e-mail at davide.cipullo@nek.uu.se. My office is **E434** at the Department of Economics. Good luck!

1. **Social choice.** According to Arrow impossibility theorem, it is impossible for a society to aggregate individual preferences in a Social Welfare Function that satisfies simultaneously the three axioms you have been introduced to in class (Universality of domain; Weak Pareto principle and Independence of irrelevant alternatives).

   (a) Was the statement correct and complete? Motivate your answer in one short paragraph.

   The statement was incomplete. Arrow's impossibility theorem instead states that it is impossible for a society to aggregate individual preferences in a Social Welfare Function that satisfies simultaneously the three axioms **and it is not dictatorial**.

   (b) Formally define the IIA axiom. Provide an intuition on and explain why the majoritarian voting rule does not satisfy it in general if there are at least three choices.

   Definition of the axiom: The Social Choice Mechanism should base its ranking of a relative to b only on the relative individual ranking of a to b, and not relative to other alternatives. Whether for individuals it is the case that $a \succ b \succ c$ or $c \succ a \succ$, the society will rank $a \succ^S b$. **The intuition relates closely to the definition**. The introduction of a new alternative can not change how the society ranks a relative to b. **To explain that the majoritarian voting rule does not satisfy the axiom in general**, I present the following example. Suppose that in a society, 40% of individuals rank $a \succ b \succ c$; 35% of them rank $b \succ c \succ a$ and 30% rank $c \succ b \succ a$. If we compare only a and b, then $b \succ^S a$, as 65% of individuals do prefer b over a. However, if we introduce the choice c, then $a \succ^S b$, as 40% of individuals would choose a, compared to the 30% that would choose b. Specifically, IIA is not satisfied in majoritarian voting because only the most preferred choice among the competing ones is revealed by individuals, who do not reveal their complete ranking.

2. **Social choice.** In this question, you will go through an application of real-world voting rules on sport competition.

Consider a pentathlon contest in which four athletes have to compete in the following disciplines: shooting, swimming, fencing, high-jumping and cross-country running. In shooting, contestants have to hit 50 marks. In swimming, they have to swim 50m as fast as possible. In fencing, they play a tournament with semi-finals and finals. In high-jumping, they have to jump over obstacles posed at an increasing height. In cross-country running, they have to run 20km.
Look at results in Table 1.

Table 1: Results of the pentathlon competition

Athlete	*Shooting*	*Swimming*	*Fencing*	*High-Jumping*	*Running*
A	43/50	1:00	#4	2.30 m	1:00:35
B	38/50	0:56	#2	2.24 m	0:58:37
C	45/50	0:55	#3	2.00 m	1:00:00
D	21/50	0:48	#1	2.09 m	1:02:40

Disciplines are different and hard to compare. In this task, you are the director of the pentathlon, and want to decide a classification mechanism. For this purpose, you make some comparisons between decision-making mechanisms:

(a) The winner of the pentathlon is the athlete who wins the largest number of disciplines. Who wins the pentathlon in this case? Explain your reasoning.

> A wins high jumping. B wins running. C wins Shooting. D wins Swimming and Fencing. **Hence, D wins the pentathlon**.

(b) If one of the athletes wins in more than half of disciplines, she wins the pentathlon. Otherwise, you compare, among the two contestants with the largest number of victories (or of other positioning, in case of a tie), their relative ranking in all five disciplines. Who wins the pentathlon in this case? Explain your reasoning.

> As shown above, D has two victories, while all other athletes have 1. Hence, we compare other positions to find D's challenger during the second round. A is second in shooting, B is second in fencing and high jumping. C is second in Swimming and running. We have to go on by comparing third positioning of B and C. B is the third classified in shooting and swimming, while C is third in fencing. Hence, **B and D compete in the second round**. B beats D in three disciplines out of five. Therefore, **B wins the pentathlon**.

(c) You are now allowed to give, for each discipline, one point to the third classified, two points to the second classified and three points to the first, and to sum results from the five disciplines. Then, you exclude the athlete with the worst score. Among the three not excluded, you assign for each discipline one point to the second classified, and two points to the first. You exclude the athlete with the worst score. Among the two remaining, the winner is the one that classifies better than the other at least in three disciplines. Who wins the pentathlon in this case? Explain your reasoning and report the tables with the scores. *Hint: Everytime you exclude an athlete, reset the score to zero for all remainers.*

> **A is excluded** because she is the athletes with the fewest points. Calculating again in the absence of A, **C is now excluded**. Among B and D, B beats D in three of the five disciplines. **Therefore, B wins the pentathlon**. Results are summarized in the next tables:

All athletes are included

Athlete	*Shooting*	*Swimming*	*Fencing*	*High-Jumping*	*Running*	***Total***
A	2	0	0	3	1	6
B	1	1	2	2	3	**9**
C	3	2	1	0	2	**8**
D	0	3	3	1	0	**7**

After the exclusion of A

Athlete	*Shooting*	*Swimming*	*Fencing*	*High-Jumping*	*Running*	***Total***
B	1	0	1	2	2	**6**
C	2	1	0	0	1	4
D	0	2	2	1	0	**5**

(d) The three decision-making procedures described above are examples of three majoritarian voting rules implemented in certain countries for national elections. The first one is the *plurality rule* (also known as *first-past-the-post*), the second one is the *runoff system* (also known as *dual-ballot rule*), and the third one is known as *elimination runoff.* Give an intuition about why they can potentially lead to different winners in the electoral competition, and report, for each of them, an empirical example of country in which the system is implemented. What is the minimum number of available choices to make each of them admit a different winner?

> The three systems can potentially lead to different winners in the electoral competition **because they differ in the order of comparison among multiple choices**. The one that turns out to be the second when N candidate runs may win under the runoff rule if she faces in the second round a challenger who she can beat in a pairwise competition. Under the elimination runoff, also the second-to-last candidate may win if she pass through several eliminations and then faces in the final round a challenger who she can beat in a pairwise competition. Plurality is e.g. implemented in the USA; runoff system is e.g. implemented in France and elimination runoff is implemented in Australia. **The minimum number of available choices to admit a different winner under each system is 4** (recall that with three candidates, the elimination runoff collapses to a runoff system).

(e) Compare your answers from (b) and (c) in view of the Condorcet Voting paradox. Limit your answer to one page.

> *This is an essay question. There is no unique suggested answer.*

3. **Voting.** Consider an economy of five individuals $i = \{1, 2, 3, 4, 5\}$ that make decisions with majority voting on how much to invest in public education. All individuals have the same quasi-linear utility function: $U_i(c_i, G) = c_i + ln(G)$ where c is a consumption (private) good, and G is the public good (education). The public good is financed through a proportional income tax t. Hence, the government budget constraint is $G = \sum_{i=1}^{5} tw_i$ while individual budget constraint is $c_i = (1-t)w_i \ \forall i \in \{1, ..5\}$. Wages are exogenously given: $w_1 = w_2 = 1$; $w_3 = 2$; $w_4 = 3$; $w_5 = 5$.

   (a) Who is the median voter in this society? Do the assumptions of the Median Voter Theorem apply? Discuss.

The median voter in this society is **individual 3**. **Assumptions of the median voter theorem hold**, as the society follows majority voting and individual preferences are complete, continuous and single-peaked.

(b) What is the expenditure on education that the society will choose? What is the tax rate? Solve the model.

Given that the assumptions of the median voter theorem hold, we know that she will be the decisive one, and the society will implement her preferred policy. Hence, we only have to solve the maximization problem for individual 3, whose wage is $w_3 = 2$. The median voter maximizes her utility with respect to c and G subject to her individual budget constraint and the government budget constraint.

Individual 3 maximizes:

Solution of the median voter problem

$$\max_{c_3,G} \quad U_3(c_3,G) = c_3 + ln(G)$$
$$\text{subject to} \quad c_3 = 2(1-t)$$
$$G = 12t$$

To simplify the problem, we substitute the two constraints in the objective function. So the problem becomes:

$$\max_t \quad U_3(t) = 2(1-t) + ln(12t)$$

The objective function is strictly concave in t, so the FOC will identify the unique maximum.

The first order conditions are:

$$\frac{\partial U_3}{\partial t} = -2 + \frac{1}{t} = 0$$

which imply $t^* = \frac{1}{2}$. Substituting into the government budget constraint, then $G^* = 6$.

Solution of the median voter problem

(c) What would have been the expenditure in education and the tax rate in the case of a benevolent central planner that maximizes an utilitarian SWF? Solve the model.

The social planner maximizes the sum of individual utilities subject to the individual constraints and to the government budget constraint. Formally,

Solution of the Social planner's problem

$$\max_{c_1..c_5,G} \quad SWF(c_1..c_5,G) = \sum_{i=1}^{5} U_i(c_i,G)$$
$$\text{subject to} \quad G = 12t$$
$$U_i(c_i,G) = c_i + ln(G)$$
$$c_i = w_i(1-t)$$

$\forall i \in \{1,...5\}$. We start as above by substituting the constraints into each individual objective functions. Hence,

$$U_i(t) = w_i(1-t) + ln(12t)$$

$\forall i \in \{1,...5\}$. So, the social planner's problem collapses to:

$$\max_{c_1..c_5,G} \quad SWF(c_1..c_5,G) = 5ln(12t) + 12(1-t)$$

The objective function is strictly concave, so FOCs identify the unique minimum point. The first order conditions are:

$$\frac{\partial SWF}{\partial t} = \frac{5}{t} - 12 = 0$$

which imply $t^* = \frac{5}{12}$. Substituting into the government budget constraint, then $G^* = 5$.

Solution of the Social planner's problem

Suppose now that all 5 individuals have equal wage $w = 2$, but preferences are instead represented by the utility function $U_i(c_i, G) = c_i^{i/5} G^{(5-i)/5}$ $\forall i \in \{1, ..5\}$. The government budget constraint and the individual budget constraints are the same as before.

(d) What is the expenditure in education that the society will choose? What is the tax rate? Solve the model.

One should notice that now all individuals have the same wage, but different relative taste for the public good compared to the private one. Hence, we can again rank them on a one-dimension scale and find the median voter, that trivially is individual 3. As in point (b), we solve the problem only for her. The median voter maximizes her utility with respect to c and G subject to her individual budget constraint and the government budget constraint.

Individual 3 maximizes:

Solution of the median voter problem (Cobb Douglas utility)

$$\max_{c_3, G} \quad U_3(c_3, G) = c_3^{\frac{3}{5}} G^{\frac{2}{5}}$$

$$\text{subject to} \quad c_3 = 2(1-t)$$

$$G = 10t$$

To simplify the problem, we substitute the two constraints in the objective function. So the problem becomes:

$$\max_t \quad U_3(t) = [2(1-t)]^{\frac{3}{5}} (10t)^{\frac{2}{5}}$$

The objective function is strictly concave in t, so the FOC will identify the unique maximum.

The first order conditions are:

$$\frac{\partial U_3}{\partial t} = -\frac{6}{5}\Big[2(1-t)\Big]^{-\frac{2}{5}} (10t)^{\frac{2}{5}} + \frac{20}{5}(10t)^{-\frac{3}{5}} \Big[2(1-t)\Big]^{\frac{3}{5}} = 0$$

$$\frac{2}{5}(10t)^{-\frac{3}{5}} \Big[2(1-t)\Big]^{-\frac{2}{5}} \Big[-3(10t) + 10(2-2t)\Big] = 0$$

$$-30t + 20 + 20t = 0$$

which implies $t^* = \frac{2}{5}$. Substituting into the government budget constraint, then $G^* = 4$.

Solution of the median voter problem (Cobb Douglas utility)

(e) What is the preferred tax rate of individual 5? Carefully explain in words and graphically the economic intuition.

It is preferable to answer the question without analitically solve the problem. Individual 5 has linear utility in consumption, and does not value G. On the one hand, taxation reduces her budget to pay for the unique good that gives her utility. Therefore, **her preferred tax rate is** $t = 0$.

(f) Compare results from points (b) and (c). Limit your answer to half page.

In (b) we got $t^* = \frac{1}{2}$ and $G^* = 6$, while in (c) we obtained $t^* = \frac{5}{12}$ and $G^* = 5$. We notice two facts: first, the solutions are different, as the preferences of the median voter are the ones that prevail in a vote under majority voting, but do not represent the preferences of the society as a whole. Hence, we do not obtain a efficient provision of public goods under the majoritarian voting. Ex-ante we do not know whether we obtain over provision or under provision, as it depends on individual preferences. In this exercise, majority voting leads to over provision of public goods compared to the social-planner solution, that represent the efficient benchmark.

4. **Bargaining in Legislature.** In a parliamentary democracy, three parties are in the House of Representativeness. We refer to them as S (*Social-democrats*), C (*Conservatives*) and P (*Populists*). None of them got the majority during the elections, but a coalition of two is necessary and sufficient to form a government. Parties can bargain over the number of ministries assigned to each party s_i, over taxation rates ($t \in [0,1]$.), or over both dimensions. A government must be composed by 5 ministries and all parties of the coalition must be represented with at least one minister. S and C are both policy and office motivated, with utility function $U_i = m_i - (a_i - 3t)^2$ where m_i is the number of ministries assigned to party i and a_i is a parameter measuring the preference for taxation, such that $a_i = \frac{3}{2}$ for S and $a_i = 1$ for C. On the other hand, the populist party P values less the number of ministries, so its utility function is $U_P = \frac{1}{3}m_P - (1-3t)^2$. If either S or C is indifferent between making the coalition with the other, or making it with P, we assume that it will do not chose P.
The share of seats is the following: $w_C > w_S > w_P$. The party with the largest share is allowed to make the first proposal. If a coalition is formed, parties receive payoffs accordingly. Otherwise, the party with the second seat share make a proposal. If a coalition is formed, parties receive payoffs accordingly. If no coalition is made, the third party makes a proposal. If again no coalition is formed, all parties get an exogenous payoff of 0.

   (a) What is the preferred tax rate of each party?

   Taxation enters negatively in the utility function, since that it reduces parties' utility. We can solve parties' problem disregarding for the moment the number of ministries, as in terms of taxation it only represents a monotonic transformation of the utility function. The function is strictly concave, so FOC will identify the unique interior maximum. Each party solves

   Solution of the party problem

$$\max_t \quad U_i(t) = -(a_i - 3t)^2$$

   First order condition w.r.t. t is:

$$6(a_i - 3t) = 0$$

   which implies $t^* = \frac{a_i}{3}$. Since we know that $a_S = \frac{3}{2}$ and $a_C = a_P = 1$, then S prefers $t = \frac{1}{2}$, while C and P prefer $t = \frac{1}{3}$.

   Solution of the party problem

   (b) Solve for the equilibrium tax rate and ministry allocation. Which parties will form the governing coalition? Can you assess at which round the proposal will be accepted?

   We solve this dynamic game by backward induction. Hence, we start from what would happen in the last round. In the last round, P would propose to C to make a coalition. as the two have the same preferences in terms of ministries. However, it cannot offer to C its continuation value, as it is constrained to give them at least one minister. Hence, P will propose to C to form a coalition in which P gets 4 ministries, P gets 1 ministry and tax rate is $t = \frac{1}{3}$. In the second round, S make an offer. S have higher expected payoff by making a coalition with C, that has a lower continuation value than P. Specifically, S has to compare a proposal in which she sets $t = \frac{1}{2}$ and $m_S = 3$ with a proposal in which she sets $t = \frac{1}{3}$ and $m_S = 4$. We compare now the two levels of utility for S:

   Solution of S problem s.t. utility of C

$$U_S(m_S = 3, t = \frac{1}{2}) = 3$$

   while

$$U_S(m_S = 4, t = \frac{1}{3}) = 4 - \frac{1}{4} = \frac{15}{4} > 3$$

Clearly, S will propose the second allocation to C. C is indifferent between this coalition and the one with the populists, so it will make a coalition with S. Now we analyze the first round. **Notice that this implies that the third stage would never be reached, hence the continuation value of P becomes zero**. In the first round, C can propose a coalition to P in which the former gets 5 ministries and the latter gets zero. Hence, the equilibrium tax rate will be $t = \frac{1}{3}$, with $m_C = 4$ and $m_P = 1$.

Solution of S problem s.t. utility of C

(c) Now assume that all parties have utility function $U_i = m_i - (\frac{1}{3} - 3t)^2$ and that a coalition can be formed even if one of the members receives 0 ministers. Solve for the equilibrium tax rate and ministry allocation. Compare with (b).

Now all parties are equal. We solve again the game by backward induction. In the last period, P would propose to either S or C to make a coalition, in which she takes all ministries and the partner gets nothing. The partner will accept as the continuation value is 0. In the second period, for S it is optimal to propose a partnership to C, in which the former gets 5 ministries and the latter gets nothing. C would accept as her continuation value is zero. **Notice that this implies that the third stage would never be reached, hence the continuation value of P becomes zero**. In the first round, C can propose a coalition to P in which the former gets 5 ministries and the latter gets zero. Hence, the equilibrium tax rate will be $t = \frac{1}{3}$ and C will obtain all ministries.

(d) Discuss your results in relation to agenda setting power. Limit your answer to half page.

See (c).